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Conditional distributions of Mandelbrot-Van Ness fractional Lévy processes and continuous-time ARMA-GARCH-type models with long memory

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Abstract

Long memory effects can be found in different kind of data from finance to hydrology. Therefore, models which can reflect these properties have become more popular in recent years especially in the fields of time series analysis, econometrics and financial mathematics. Mandelbrot-Van Ness fractional Lévy processes allow for such stationary long memory effects in their increments and have been used in many settings ranging from fractionally integrated continuous-time (autoregressive) moving average processes and exponential GARCH models to general stochastic differential equations. However, their conditional distributions have not yet been considered in detail. In this paper, we provide a closed formula for their conditional characteristic functions and suggest several applications to continuous-time ARMA-GARCH-type models with long memory.

Keywords: fractional Lévy processes, Mandelbrot-Van Ness kernel, long memory, conditional characteristic function, prediction, forecasting, FICARMA, FIECOGARCH

1. Introduction

Fractional Lévy processes can be obtained in several ways via convolution of classical Lévy processes and have been introduced to allow for long memory effects in increments. There exist several possibilities to choose these convolution integrands with Mandelbrot-Van Ness and Molchan-Golosov kernels being the most prominent examples. However in contrast to Brownian motion, both of these approaches do not lead to the same kind of fractional processes. Mandelbrot-Van Ness fractional Lévy processes (MvN-fLps) which have been considered by Marquardt (2006) allow for stationary increments, while Molchan-Golosov fractional

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Lévy processes (MG-fLps) offer the possibility of having fractional subordinators, ie. strictly increasing processes (cf. Bender and Marquardt (2009), Tikanmäki and Mishura (2011) or Fink (2013)). While the later ones could be a good choice for derivative pricing in eg. interest rate, credit or stochastic volatility settings where positive processes are needed, the first ones might be better suited for time series analysis due to their stationarity (eg. Haug and Czado (2010)). In all these situations however, whether one is interested in time series prediction or derivative pricing, it is very useful to understand the respective conditional distributions. While these have been considered already in Fink (2013) for MG-fLps, to the best of our knowledge, there is not yet a full characterization for MvN-fLps in the literature. In this paper we want to close this gap by providing a closed formula for the conditional characteristic function of these processes which completely characterizes the conditional distribution and allows for easy and quick calculations of moments like conditional expectation and conditional variance via differentiation.

The paper is structured as follows. Section 2 provides a brief review on MvN-fLps and states the necessary integration-concept and a linkage between the MvN-fLp and its driving Lévy process. Section 3 provides the main theorem about conditional distributions based on the conditional characteristic functions. For better readability, its lengthy proof is postponed to the end of the paper. In Section 4 we shall discuss several possible applications of our prediction formula. Finally, Section 5 contains the proof of our main theorem followed by a brief outlook to future research.

For the rest of the paper, we will always work on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore we impose that $L = (L(t))_{t\in\mathbb{R}}$ is a given two-sided zero-mean Lévy process with finite second moments, ie. $E[L(1)^2] < \infty$. Its (augmented) filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ is assumed to satisfy the usual conditions of right-continuity and completeness (cf. Theorem 2.1.9 of Applebaum (2004)). The characteristic function of L shall be given for $u \in \mathbb{R}$ by

$$E[\exp\{iuL(t)\}] = \exp\{t\psi(u)\}, \quad t \in \mathbb{R}$$

with

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (\exp\{iux\} - 1 - iux\mathbf{1}_{\{|x|<1\}})\nu(dx),$$

where $\gamma \in \mathbb{R}$, $\sigma^2 > 0$ and the measure ν satisfies

$$\nu(\{0\}) = 0, \quad \int_{\mathbb{R}^n} (\|x\|^2 \wedge 1)\nu(dx) < \infty.$$

The indicator function shall be denoted by $\mathbf{1}_{[s,t)}$ for $s, t \in \mathbb{R}$ while we set

$$\mathbf{1}_{[s,t)} = -\mathbf{1}_{(t,s]}$$

when $t < s$.

2. Mandelbrot-Van Ness fractional Lévy processes

In this section we want to provide a brief review on Mandelbrot-Van Ness fractional Lévy processes. We want to stress that although our main focus lies in long memory settings ($d \in (0, \frac{1}{2})$), our considerations shall include the perturbance case ($d \in (-\frac{1}{2}, 0)$) as well.

In the following, we shall heavily draw from Marquardt (2006), state some straightforward extensions to the situation of $d \in (-\frac{1}{2}, 0)$ and concentrate on the necessary concepts needed for the conditional characteristic functions later in Section 3. In contrast to Marquardt (2006), we shall also allow our driving Lévy process to have a Brownian part but as fractional Brownian motion already has been investigated in much detail, this generalization is straightforward having the Lévy-Itô-decomposition in mind.

Definition 2.1 (Version of Marquardt (2006), Definition 3.1, and Tikanmäki and Mishura (2011), Definition 2.3). *For $d \in (-\frac{1}{2}, \frac{1}{2})$ set*

$$M_d(t) := \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\infty} [(t-s)_+^d - (-s)_+^d] L(ds), \quad t \in \mathbb{R}, \tag{2.1}$$

where the integrals are considered in the $L^2(\Omega)$-sense. Then we call the process $M_d = (M_d(t))_{t \in \mathbb{R}}$ a Mandelbrot-Van Ness fractional Lévy process (MvN-fLp) *and L the* driving Lévy process *of M_d.*

The next proposition summarizes the main properties of MvN-fLps.

Proposition 2.2 (Version of Marquardt (2006), Theorem 3.7, Theorem 4.1 and Theorem 4.4). *For $d \in (-\frac{1}{2}, \frac{1}{2})$ let M_d be a MvN-fLp. Then we have:*

(i) If $d \in (0, \frac{1}{2})$, then M_d has a modification with continuous sample paths.

(ii) For $s, t \in \mathbb{R}$

$$\mathbb{C}\mathrm{ov}(M_d(t), M_d(s)) = \frac{E[L(1)]^2}{2\Gamma(2d+2)\sin(\pi(d+\frac{1}{2}))} [|t|^{2d+1} + |s|^{2d+1} - |t-s|^{2d+1}].$$

(iii) M_d has stationary increments and is symmetric.

To define integration with respect to MvN-fLps, we need the *(right-sided) Riemann-Liouville fractional integral* for $d \in (0, \frac{1}{2})$ defined by

$$I_-^d[g](x) = \frac{1}{\Gamma(d)} \int_x^{\infty} g(t)(t-x)^{d-1} dt, \quad x \in \mathbb{R},$$

if the integrals exist almost everywhere. Furtheremore the *(right-sided) Marchaud fractional derivative* is introduced as a somewhat inverse operator using

$$\mathbf{D}^d_-[g](x) = \lim_{\varepsilon \searrow 0} \mathbf{D}^d_{-,\varepsilon}[g](x)$$

if the limit exists almost everywhere for $x \in \mathbb{R}$, where

$$\mathbf{D}^d_{-,\varepsilon}[g](x) = -\frac{d}{\Gamma(1-d)} \int_\varepsilon^\infty \frac{g(x) - g(x+t)}{t^{1+d}} dt, \quad x \in \mathbb{R}, \quad \varepsilon > 0,$$

The expression $\mathbf{D}^d_{-,\varepsilon}$ is also often called *truncated Marchaud fractional derivative.* A very detailed survey of these (and related concepts) can be found in Samko, Kilbas and Marichev (1993). Especially the question of existence of these operators is rather sophisticated; however in this paper, we shall only consider specific situations where this existence is ensured.

For negative $d \in (-\frac{1}{2}, 0)$ we shall set $I^d_- := \mathbf{D}^{-d}_-$ and $\mathbf{D}^d_- := I^{-d}_-$ while for $d = 0$ we define $I^d_- := \mathrm{id} =: \mathbf{D}^d_-$.

In the light of Pipiras and Taqqu (2000; 2001) we construct similar to Marquardt (2006) the following integrand space: define for $d \in (0, \frac{1}{2})$ $\tilde{H}^d$ as the completion of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with respect to the norm

$$\|g\|_{\tilde{H}^d} := \left(\mathbb{E}[L(1)^2] \int_{\mathbb{R}} (I^d_-[g](v))^2 dv \right)^{\frac{1}{2}}.$$

As pointed out by Bender and Elliott (2003), multiplication by indicator functions can actually increase this norm. To avoid any such issues, we shall consider an adjusted integrand space defined by

$$H^d := \{g : \mathbb{R} \to \mathbb{R} | \forall -\infty \leq a \leq b \leq \infty : \mathbf{1}_{[a,b)} g \in \tilde{H}^d\}.$$

By Proposition 5.1 of Marquardt (2006) it follows that

$$L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \subseteq H^d$$

which shows that the restricted space is still rich enough for many applications. For $d = 0$ just set $H^d = L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

Although the focus of this paper lies on long memory models, we still want to include the perturbance case: for $d \in (-\frac{1}{2}, 0)$ define

$$H^d := \left\{ g : \mathbb{R} \to \mathbb{R} | \exists \phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) : g = I^{-d}_-[\phi] \right\}.$$

Here one can actually show that this space is closed with respect to multiplication with indicator functions: if $g \in H^d$ with $g = I_-^{-d}[\phi]$, then for all $-\infty \le a \le b \le \infty$, we have

$$\mathbf{1}_{[a,b)} g = I_-^{-d}[\mathbf{D}_-^{-d}[\mathbf{1}_{[a,b)} I_-^{-d}[\phi]]]$$

The argument is quite lengthy but very similar to the proof of Proposition 5.1. Since the focus of this paper are long memory models, we leave it to the interested reader.

Theorem 2.3. *[Extension of Marquardt (2006), Theorem 5.3] For $d \in (-\frac{1}{2}, \frac{1}{2})$ let M_d be a MvN-fLp and $g \in H^d$. Then the integral $\int_{\mathbb{R}} g(v) M_d(dv)$ exists as $L^2(\Omega)$-limit of approximating step functions. Furthermore*

$$\int_{\mathbb{R}} g(v) M_d(dv) \overset{L^2(\Omega)}{=} \int_{\mathbb{R}} I_-^d[g](v) L(dv).$$

Proof. For $d \in (0, \frac{1}{2})$ this is covered by Theorem 5.3 of Marquardt (2006) and Theorem 3.2 of Pipiras and Taqqu (2000) while the case $d \in (-\frac{1}{2}, 0)$ follows similarly having Theorem 3.3 of Pipiras and Taqqu (2000) in mind. Finally, the situation $d = 0$ is just the classical Lévy integration. □

3. Conditional distribution of MvN-fLps

Let $\mathcal{F}^{M_d} = (\mathcal{F}_t^{M_d})_{t \in \mathbb{R}}$ and $\mathcal{F}^L = (\mathcal{F}_t^L)_{t \in \mathbb{R}}$ be the (augmented) filtration generated by M_d and its underlying Lévy process L, respectively. Therefore

$$\mathcal{F}_t^{M_d} = \sigma\overline{\{M_d(s), s \in (-\infty, t]\}}, \quad \mathcal{F}_t^L = \sigma\overline{\{L(s), s \in (-\infty, t]\}}.$$

In contrast to MG-fLps which are defined via convolution on compact time intervals, the filtration generated by a MvN-fLp is in general not equal to the one generated by its driving Lévy process, ie. $\mathcal{F}^{M_d} \neq \mathcal{F}^L$. This can be seen by the following argument: as a consequence of our Theorem 2.3, Theorem 6.1 of Samko, Kilbas and Marichev (1993) and Lemma 2 of Pipiras and Taqqu (2002), we obtain the following representations in the $L^2(\Omega)$-sense:

$$M_d(t) = \int_{\mathbb{R}} I_-^d[\mathbf{1}_{[0,t)}](v) L(dv), \quad L(t) = \int_{\mathbb{R}} \mathbf{D}_-^d[\mathbf{1}_{[0,t)}](v) M_d(dv), \quad t \in \mathbb{R}.$$

Since for $t \ge 0$ we have

$$\operatorname{supp}(I_-^d[\mathbf{1}_{[0,t)}]) \cap (t, \infty] = \emptyset,$$

it follows that

$$\mathcal{F}_t^{M_d} = \mathcal{F}_t^L.$$

However on the other side

$$\operatorname{supp}(I_-^d[\mathbf{1}_{[0,t)}]) = \operatorname{supp}(I_-^d[\mathbf{1}_{(t,0]}]) = (-\infty, 0)$$

for all $t < 0$. Therefore the filtrations are not equal for such $t < 0$, similar to the case of fractional Brownian motion (cf. Proposition 1 of Pipiras and Taqqu (2002)). It might be more natural to condition on $\mathcal{F}^{M_d}$, especially when having conditional distributions in mind which are - as it is well known - completely specified by the conditional characteristic functions. However in our following main theorem we will cover both cases. Its proof will be postponed to Section 5.

Theorem 3.1. *For $d \in (-\frac{1}{2}, \frac{1}{2})$ let M_d be a MvN-fLp and $g \in H^d$. Then we have for all $s, t \in \mathbb{R}$ with $s \leq t$ and $u \in \mathbb{R}$*

$$\begin{aligned} &\mathbb{E}\Big[\exp\Big\{iu\int_{-\infty}^{t} g(v)M_d(dv)\Big\}\Big|\mathcal{F}_s^{M_d}\Big] \\ \overset{(i)}{=}\ &\exp\Big\{iu\int_{-\infty}^{s} g(v)M_d(dv) + iu\int_{-\infty}^{s} \mathbf{D}_-^d[\mathbf{1}_{(-\infty,s)} I_-^d[\mathbf{1}_{[s,t)}g]](v)M_d(dv)\Big\} \\ &\times \exp\Big\{\int_s^t \psi\{u \cdot I_-^d[\mathbf{1}_{[s,t)}g](v)\}dv\Big\} \\ \overset{(ii)}{=}\ &\mathbb{E}\Big[\exp\Big\{iu\int_{-\infty}^{t} g(v)M_d(dv)\Big\}\Big|\mathcal{F}_s^{L}\Big] \end{aligned}$$

while

- *equation (i) holds in the $L^2(\Omega)$-sense for $s \geq 0$ and in distribution for $s < 0$,*
- *equation (ii) holds in the $L^2(\Omega)$-sense for all $s \in \mathbb{R}$.*

Furthermore, we have for $z \in \mathbb{R}$

$$\mathbf{D}_-^d[\mathbf{1}_{(-\infty,s)} I_-^d[\mathbf{1}_{[s,t)}g]](z) = \mathbf{1}_{(-\infty,s)}(z)\frac{\sin(\pi d)}{\pi}(s-z)^{-d}\int_s^t g(\eta)\frac{(\eta - s)^d}{\eta - z}d\eta.$$

4. Applications to fractional time series models and stochastic differential equations

In this section we want to present some potential applications of Theorem 3.1 to stress its importance. We shall focus especially on the forecasting problem in continuous-time fractional ARMA-GARCH-type models. For the rest of this section, let $d \in (0, \frac{1}{2})$ and consider only MvN-fLps without a (fractional) Brownian component.

4.1. Fractionally integrated moving average processes

Motivated by Theorem 6.2 of Marquardt (2006), fractionally integrated moving average (FIMA) processes can be introduced directly via a moving average representation with respect MvN-fLps.

Definition 4.1. *For $d \in (0, \frac{1}{2})$ let M_d be a MvN-fLp and $g \in H^d$ with $g(t) = 0$ for all $t < 0$. Then the respective* FIMA process $Y_d = (Y_d(t))_{t\in\mathbb{R}}$ *is given by*

$$Y_d(t) = \int_{-\infty}^{t} g(t-s)M_d(ds), \quad t \in \mathbb{R}.$$

Similar to MvN-fLps, these kind of processes allow to model long range dependence in the context of time series analysis (Marquardt (2006), Theorem 6.3). Theorem 3.1 now provides the necessary tool to calculate forecasts in such a setting. After estimating the kernel function g, we can obtain predictions for future values, via the following:

Theorem 4.2. *Let $Y_d = (Y_d(t))_{t\in\mathbb{R}}$ be a FIMA process such that $\mathcal{F}^{Y_d} = \mathcal{F}^{M_d}$. Then we have*

$$\mathbb{E}[Y_d(t)|\mathcal{F}_s^{Y_d}] \overset{(i)}{=} \int_{-\infty}^{s} g(t-v)M_d(dv) + \int_{-\infty}^{s} \mathbf{D}_-^d[\mathbf{1}_{(-\infty,s)}I_-^d[\mathbf{1}_{[s,t)}g(t-\cdot)]](v)M_d(dv),$$

while equation (i) holds in the $L^2(\Omega)$-sense for $s \geq 0$ and in distribution for $s < 0$.

Proof. By Theorem 3.1 we already know the complete conditional characteristic function of Y_d. The forecast can then be derived by differentiation. For $s \geq 0$ we have

$$\begin{aligned}\mathbb{E}[Y_d(t)|\mathcal{F}_s^{Y_d}] &= \mathbb{E}\Big[\int_{-\infty}^{t} g(t-v)M_d(dv)\Big|\mathcal{F}_s^{M_d}\Big] \\ &= -i\frac{\partial}{\partial u}\Big|_{u=0}\mathbb{E}\Big[\exp\Big\{iu\int_{-\infty}^{t} g(t-v)M_d(dv)\Big\}\Big|\mathcal{F}_s^{M_d}\Big] \\ &= \int_{-\infty}^{s} g(t-v)M_d(dv) + \int_{-\infty}^{s} \mathbf{D}_-^d[\mathbf{1}_{(-\infty,s)}I_-^d[\mathbf{1}_{[s,t)}g(t-\cdot)]](v)M_d(dv).\end{aligned}$$

The situation $s < 0$ works in a similar way but only holds in distribution. □

4.2. Fractionally integrated CARMA processes

An important special case of FIMA processes is the class of fractionally integrated continuous-time autoregressive moving average (FICARMA) processes initially derived by Brockwell (2004) via a fractional integration kernel with respect to a Lévy process L. However, as the core object of this paper are MvN-fLps, we shall state the FICARMA-definition similar to Marquardt (2007).

Definition 4.3 (Univariate version of Marquardt (2007), Definition 3.4)**.** *For $d \in (0, \frac{1}{2})$ let M_d be a MvN-fLp. Furthermore, let a and b be polynomials with*

$$a(z) = z^p + a_1 z^{p-1} + \ldots + a_p, \quad \text{and} \quad b(z) \quad = \quad b_0 z^q + b_1 z^{q-1} + \ldots + b_q,$$

where $p > q$ are integers, $a_p \neq 0$, $b_0 \neq 0$, $b_q \neq 0$ and the zeros of a are assumed to lie in $(-\infty, 0) + i\mathbb{R}$. Then a FICARMA$(p, d, q)$ process $Y_d = (Y_d(t))_{t\in\mathbb{R}}$ *is given by*

$$Y_d(t) = \int_{-\infty}^{t} g(t-s) M_d(ds), \quad t \in \mathbb{R},$$

with

$$g(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\theta t} \frac{b(i\theta)}{a(i\theta)} d\theta, \quad t \in \mathbb{R}.$$

As FICARMA processes are nested within the FIMA setting and one can show $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, Theorem 4.2 can directly be applied.

4.3. Stochastic volatility and fractional GARCH-type settings

As pointed out on several occasions in the literature, the existence of a natural continuous-time fractional version of the discrete-time stationary GARCH process is not clear (cf. Haug and Czado (2010) or Mikosch and Stărică (2002)). This is eg. reflected by the fact that MvN-fLps have stationary increments but it is not possible to obtain a strictly positive process - which one would like to have in a GARCH setting. There are several possibilities to tackle this issue: for example, one could use fractional MG-fLp subordinators (cf. Bender and Marquardt (2009) for a stochastic volatility model based on these processes) which however do not have stationary increments. Another approach is investigated by the recent model of Haug, Klüppelberg and Straub (2014) which is based on a modified MvN-kernel but generally leaves the MvN-setting.

Finally, one can always transform MvN-fLps to make them positive. Eg. Haug and Czado (2010) introduced such a general model using an exponential FIMA process:

Definition 4.4 (Version of Haug and Czado (2010), Definition 2.1)**.** *Let K be a Lévy process with zero-mean and unit variance. Furthermore take L as the Lévy process constructed from K via equation (2.7) of Haug and Czado (2010). For $d \in (0, \frac{1}{2})$ let M_d be a MvN-fLp driven by L.*

We assume given polynomials a and b with

$$a(z) = 1 - a_1 z + \ldots + a_q z^q, \quad \text{and} \quad b(z) \quad = \quad b_1 + b_2 z + \ldots + b_p z^{p-1},$$

where $q > p$ are integers, $a_q \neq 0$, $b_p \neq 0$ and the zeros of a are assumed to lie in $(-\infty, 0) + i\mathbb{R}$.

Then the fractionally integrated exponential COGARCH(p,d,q) (FIECOGARCH(p,d,q)) *is given by*

$$dG_d(t) = \sigma_d(t)dK(t), \quad t > 0, \quad G_d(0) = 0,$$

where

$$\log(\sigma_d^2(t+)) = \mu + \int_{-\infty}^{t} g(t-s)M_d(ds), \quad t > 0$$

with $\mu \in \mathbb{R}$ *and initial value* $\log(\sigma_d^2(0))$ *independent of* L. *The integrand is given by*

$$g(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\theta t}\frac{b(i\theta)}{a(i\theta)}d\theta, \quad t \in \mathbb{R}.$$

As the volatility in the FIECOGARCH(p,d,q) setting is designed as an exponential FIMA process, we can again use Theorem 4.2 for forecasting:

Theorem 4.5. *Let* $\sigma_d^2 = (\sigma_d^2(t+))_{t\geq 0}$ *be the volatility process in the FIECOGARCH(p,d,q) model and assume the existence of the respective exponential moment and* $(\mathcal{F}^{\sigma_d^2})_{t\geq 0} = (\mathcal{F}^{M_d})_{t\geq 0}$. *Then we have for* $s \geq 0$ *the following* $L^2(\Omega)$*-equality:*

$$\begin{aligned}
& \mathbb{E}[\sigma_d^2(t+)) | \mathcal{F}_s^{\sigma_d^2}] \\
= \; & \exp\left\{ iu \int_{-\infty}^{s} g(t-v)M_d(dv) + iu \int_{-\infty}^{s} \mathbf{D}_-^d[\mathbf{1}_{(-\infty,s)} I_-^d[\mathbf{1}_{[s,t)} g(t-\cdot)]](v) M_d(dv) \right\} \\
& \times \exp\left\{ \int_{s}^{t} \psi\{u \cdot I_-^d[\mathbf{1}_{[s,t)} g(t-\cdot)](v)\} dv \right\}.
\end{aligned}$$

Proof. Follows directly by invoking Theorem 3.1 and continuation of the characteristic function to $\mathbb{C}$. □

4.4. Stochastic differential equations

Finally, our results can also be of use to general stochastic differential equations (sdes). Invoking Theorem 3.1 we are able to calculate conditional characteristic functions of integrals driven by MvN-fLps. This also includes fractional Lévy Ornstein-Uhlenbeck processes of the type

$$\mathcal{M}_d^\lambda(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} M_d(ds), \quad t \in \mathbb{R}, \quad \lambda \geq 0.$$

Fink and Klüppelberg (2011) showed that these kind of processes not only exist in the pathwise sense, but are also the unique stationary and pathwise solutions to the respective pathwise Langevin sdes

$$d\mathcal{M}_d^\lambda(t) = -\lambda\mathcal{M}_d^\lambda(t)dt + M_d(dt), \quad t \in \mathbb{R}.$$

Based on earlier work of Zähle (2001) and Buchmann and Klüppelberg (2006), Fink and Klüppelberg (2011) showed that under some conditions on coefficient functions of a general sde

$$dX(t) = \mu(X(t))dt + \sigma(X(t))M_d(dt), \quad t \in \mathbb{R},$$

pathwise solutions can be obtained of the form $X(t) = f(\mathcal{M}_d^\lambda(t))$, $t \in \mathbb{R}$, where f is an invertible function. Using well-known fourier techniques, like in Theorem 3.8 of Fink, Klüppelberg and Zähle (2012), we can straightforwardly calculate the conditional characteristic function of such pathwise solutions under assumption of existing suitable exponential moments of $\mathcal{M}_d^\lambda$ via

$$\begin{aligned} & \mathbb{E}\Big[e^{iuf(\mathcal{M}_d^\lambda(t))}\Big|\mathcal{F}_s^{\mathcal{M}_d^\lambda}\Big] = \mathbb{E}\Big[e^{iuf(\mathcal{M}_d^\lambda(t))}\Big|\mathcal{F}_s^{M_d}\Big] \\ = & \int_{\mathbb{R}} \Big(\mathbb{E}\Big[e^{(i\xi+1)\mathcal{M}_d^\lambda(t)}\Big|\mathcal{F}_s^{M_d}\Big]\widehat{g_+}(\xi,u) + \mathbb{E}\Big[e^{(i\xi-1)\mathcal{M}_d^\lambda(t)}\Big|\mathcal{F}_s^{M_d}\Big]\widehat{g_-}(\xi,u)\Big)\, d\xi \end{aligned}$$

for $u \in \mathbb{R}$, $t, s \in \mathbb{R}$ with $s \le t$ and

$$\widehat{g}_\pm(\xi,u) = (2\pi)^{-1}\int_{\mathbb{R}_\pm} e^{-(i\xi\pm1)x+iuf(x)}dx, \quad \xi \in \mathbb{R}.$$

The conditional expectation

$$E[e^{(i\xi+1)\mathcal{M}_d^\lambda(t)}|\mathcal{F}_s^{M_d}]$$

is then given by the continuation of the characteristic function in Theorem 3.1 to $\mathbb{C}$.

5. Proof of Theorem 3.1

Before considering the proof of our main theorem, we need the following proposition:

Proposition 5.1. *For $d \in (-\frac{1}{2},\frac{1}{2})$, $g \in H^d$ and $t, s \in \mathbb{R}$ with $s \le t$ the function*

$$\begin{aligned} \mathbb{R} \to \mathbb{R}, \quad z \mapsto \quad & \mathbf{D}_-^d[\mathbf{1}_{(-\infty,s)}I_-^d[\mathbf{1}_{[s,t)}g]](z) \\ = \quad & \mathbf{1}_{(-\infty,s)}(z)\frac{\sin(\pi d)}{\pi}(s-z)^{-d}\int_s^t g(\eta)\frac{(\eta-s)^d}{\eta-z}d\eta \end{aligned} \tag{5.2}$$

is well defined with

$$I_-^d[\mathbf{D}_-^d[\mathbf{1}_{(-\infty,s)}I_-^d[\mathbf{1}_{[s,t)}g]]] = \mathbf{1}_{(-\infty,s)}I_-^d[\mathbf{1}_{[s,t)}g] \tag{5.3}$$

Proof. The statements are trivial for $d = 0$ since $I_-^0 = \mathbf{D}_-^0 = \mathrm{id}$ per definition. It follows that

$$\mathbf{1}_{(-\infty,s)}I_-^0[\mathbf{1}_{[s,t)}g] = \mathbf{1}_{(-\infty,s)}\mathbf{1}_{[s,t)}g \equiv 0.$$

Consider $d \in (0, \frac{1}{2})$. Since $g \in H^d$ we have $\mathbf{1}_{(-\infty,s)}I_-^d[\mathbf{1}_{[s,t)}g] \in L^2(\mathbb{R})$. If $z \geq s$ both sides of (5.3) are equal to zero. However for $z < s$ the existence of the respective Marchaud fractional derivative is not trivial. Therefore we start with its truncated version and take ε such that $z < s - \varepsilon$:

$$\begin{aligned}
& \mathbf{D}_{\varepsilon,-}^d[\mathbf{1}_{(-\infty,s)}I_-^d[\mathbf{1}_{[s,t)}g]](z) \\
= & \frac{d}{\Gamma(1-d)\Gamma(d)} \\
& \cdot \int_\varepsilon^\infty \frac{\mathbf{1}_{(-\infty,s)}(z)\big[\int_s^t \frac{g(\eta)}{(\eta-z)^{1-d}}d\eta\big] - \mathbf{1}_{(-\infty,s)}(z+\theta)\big[\int_s^t \frac{g(\eta)}{(\eta-z-\theta)^{1-d}}d\eta\big]}{\theta^{1+d}}d\theta \\
= & \frac{\sin(\pi d)d}{\pi}\Bigg\{\int_{s-z}^\infty \frac{\int_s^t \frac{g(\eta)}{(\eta-z)^{1-d}}d\eta}{\theta^{1+d}}d\theta \\
& + \int_\varepsilon^{s-z} \frac{\int_s^t g(\eta)[(\eta-z)^{d-1} - (\eta-z-\theta)^{d-1}]d\eta}{\theta^{1+d}}d\theta\Bigg\} \\
= & \frac{\sin(\pi d)d}{\pi}\Bigg\{\frac{(s-z)^{-d}}{d}\int_s^t \frac{g(\eta)}{(\eta-z)^{1-d}}d\eta \\
& \int_s^t g(\eta)\left(\int_\varepsilon^{s-z} \frac{(\eta-z)^{d-1} - (\eta-z-\theta)^{d-1}}{\theta^{1+d}}d\theta\right)d\eta\Bigg\}
\end{aligned}$$

where we achieved the last equality by Fubini's theorem. Considering just the inner integral of the second term, we obtain by substituting $\zeta = (\eta - z)/\theta$

$$\begin{aligned}
&\int_\varepsilon^{s-z} \frac{(\eta - z)^{d-1} - (\eta - z - \theta)^{d-1}}{\theta^{1+d}} d\theta \\
= \;& \int_\varepsilon^{s-z} \frac{(\eta - z)^{d-1}}{\theta^{1+d}} - \int_\varepsilon^{s-z} \frac{(\eta - z - \theta)^{d-1}}{\theta^{1+d}} d\theta \\
= \;& \frac{(\eta - z)^{d-1}}{d} \Big[\varepsilon^{-d} - (s-z)^{-d}\Big] - \int_{(\eta-z)/(s-z)}^{(\eta-z)/\varepsilon} \frac{(\zeta - 1)^{d-1}}{\eta - z} d\zeta \\
= \;& \frac{(\eta - z)^{d-1}}{d} \Big[\varepsilon^{-d} - (s-z)^{-d}\Big] - \frac{1}{(\eta - z)d} \Big[\big(\frac{\eta - z}{\varepsilon} - 1\big)^d + \big(\frac{\eta - z}{s - z} - 1\big)^d\Big].
\end{aligned}$$

Plugging everything together, we arrive at

$$\begin{aligned}
&\mathbf{D}^d_{\varepsilon,-}[\mathbf{1}_{(-\infty,s)} I^d_-[\mathbf{1}_{[s,t)} g]](z) \\
= \;& \frac{\sin(\pi d)}{\pi} \left\{ \int_s^t g(\eta) \frac{(\eta - z)^{d-1} - (\eta - z - \varepsilon)^{d-1}}{\varepsilon^d} d\eta + (s-z)^{-d} \int_s^t g(\eta) \frac{(\eta - s)^d}{\eta - z} d\eta \right\}.
\end{aligned}$$

By binomial expansion of the numerator $(\eta - z)^{d-1} - (\eta - z - \varepsilon)^{d-1}$ one can show that the first summand goes to zero for $\varepsilon \to 0$ and it follows that for $z \in \mathbb{R}$

$$\mathbf{D}^d_-[\mathbf{1}_{(-\infty,s)} I^d_-[\mathbf{1}_{[s,t)} g]](z) = \mathbf{1}_{(-\infty,s)}(z) \frac{\sin(\pi d)}{\pi} (s-z)^{-d} \int_s^t g(\eta) \frac{(\eta - s)^d}{\eta - z} d\eta.$$

It remains to show that (5.3) holds. Similar to Gripenberg and Norros (1996), we can show that $\mathbf{D}^d_-[\mathbf{1}_{(-\infty,s)} I^d_-[\mathbf{1}_{[s,t)} g]] \in L^2(\mathbb{R})$ and therefore its fractional integral exists. Take $x < s$ and after applying Fubini's theorem we obtain

$$I^d_-[\mathbf{D}^d_-[\mathbf{1}_{(-\infty,s)} I^d_-[\mathbf{1}_{[s,t)} g]]](x) = \frac{1}{\Gamma(1-d)} \int_s^t g(\eta)(\eta - s)^d \Big(\int_x^s \frac{(\xi - x)^{d-1}}{(s-\xi)^d(\eta - \xi)} d\xi \Big) d\eta.$$

Considering the right-hand-side of (5.3), it would be sufficient to show

$$\int_x^s \frac{(\xi - x)^{d-1}}{(s-\xi)^d(\eta - \xi)} d\xi = \frac{\Gamma(1-d)}{\Gamma(d)} \frac{(\eta - x)^{d-1}}{(\eta - s)^d} \tag{5.4}$$

for $x \leq s \leq \eta$. In order to do that, we shall work with the basic result from Gripenberg and Norros (1996): based on our notation, they showed for a fractional Brownian motion

B^d ($d \in (0, \frac{1}{2})$) without using fractional integration or differentiation that for $t \geq 0$

$$\begin{aligned}\mathbb{E}[B^d(t)|\mathcal{F}_0^{B^d}] &= \int_{-\infty}^0 h(t,x)B^d(dx), \quad \text{where} \\ h(t,x) &= \mathbf{1}_{(-\infty,0)}(x)\frac{\sin(\pi d)}{\pi}(-x)^{-d}\int_0^t \frac{\eta^d}{\eta - x}d\eta.\end{aligned}$$

However since $\mathcal{F}_0^{B^d} = \mathcal{F}_0^B$, one can calculate the conditional expectation in terms of the driving Brownian motion as well and obtain via Theorem 2.3 and the independent increments of B

$$\mathbb{E}[B^d(t)|\mathcal{F}_0^{B^d}] = \mathbb{E}\Big[\int_{\mathbb{R}} I_-^d[\mathbf{1}_{[0,t)}](x)B(dx)\Big|\mathcal{F}_0^B\Big] = \int_{-\infty}^0 I_-^d[\mathbf{1}_{[0,t)}](x)B(dx).$$

Due to the uniqueness of the conditional expectation, we can deduce that

$$\int_{-\infty}^0 h(t,x)B^d(dx) = \int_{-\infty}^0 I_-^d[\mathbf{1}_{[0,t)}](x)B(dx)$$

holds almost surely. Since their second moments exist, both random variables have to be equal in the $L^2(\Omega)$-sense as well. Invoking Theorem 2.3, this translates to

$$I_-^d[\mathbf{1}_{(-\infty,0)}h(t,\cdot)](x) = \mathbf{1}_{(-\infty,0)}(x)I_-^d[\mathbf{1}_{[0,t)}](x)$$

almost everywhere. Using fractional calculus and (5.2), this means that (5.3) holds for $g = \mathbf{1}_{[0,t)}$ and $s = 0$, ie.

$$\frac{1}{\Gamma(1-d)}\int_0^t \eta^d\Big(\int_x^0 \frac{(\xi - x)^{d-1}}{(-\xi)^d(\eta - \xi)}d\xi\Big)d\eta = \frac{1}{\Gamma(d)}\int_0^t (\eta - x)^{d-1}d\eta$$

for almost all $x \leq 0$. As the integrand functions on both sides are continuous and the above equality holds for all $t \geq 0$, it follows that for all $0 \leq \eta \leq t$:

$$\int_x^0 \frac{(\xi - x)^{d-1}}{(-\xi)^d(\eta - \xi)}d\xi = \frac{\Gamma(1-d)}{\Gamma(d)}\frac{(\eta - x)^{d-1}}{\eta^d}$$

For general $x \leq s \leq \eta \leq t$ however, (5.4) can always be shifted back to the above by invoking the substitution $\zeta = \xi - s$. Therefore we have shown that (5.4) holds and the assertion follows for $d \in (0, \frac{1}{2})$.

Finally, take $d \in (-\frac{1}{2}, 0)$. Then by definition we have $I_-^d[\mathbf{1}_{[s,t)}g] \in L^2(\mathbb{R})$ and therefore $\mathbf{1}_{(-\infty,s)}I_-^d[\mathbf{1}_{[s,t)}g] \in L^2(\mathbb{R})$ as well. Having in mind that I_-^d is now actually the Marchaud derivative and $\mathbf{D}^d_-$ the fractional integration operator, it follows that $\mathbf{D}^d_-[\mathbf{1}_{(-\infty,s)}I^d_-[\mathbf{1}_{[s,t)}g]]$ is well-defined. By Theorem 6.1 of Samko, Kilbas and Marichev (1993) we additionally see that (5.3) holds. The representation (5.2) is derived similar to the case $d \in (0, \frac{1}{2})$. □

Having this in mind, we shall carry out the proof of Theorem 3.1 in three steps starting with the simplest case, where one conditions on $\mathcal{F}^{M_d}_s$ with $s \geq 0$. For ease of notation, we shall always imply equality in the $L^2(\Omega)$-sense when applying Theorem 2.3 and not mention it specifically every such time.

Lemma 5.2. *For $d \in (-\frac{1}{2}, \frac{1}{2})$ let $g \in H^d$. Then for all $s, t \in \mathbb{R}^+$ with $s \leq t$ and $u \in \mathbb{R}$ equations (i) and (ii) hold true in the $L^2(\Omega)$-sense.*

Proof. Since $g \in H^d$, we know that all restrictions of g are also element of H^d by definition. We then decompose and apply Theorem 2.3 as follows:

$$\begin{aligned}
\int_{-\infty}^t g(v)M_d(dv) &= \int_{-\infty}^s g(v)M_d(dv) + \int_s^t g(v)M_d(dv) \\
&= \int_{-\infty}^s g(v)M_d(dv) + \int_{\mathbb{R}} \mathbf{1}_{[s,t)}(v)g(v)M_d(dv) \\
&= \int_{-\infty}^s g(v)M_d(dv) + \int_{\mathbb{R}} I^d_-[\mathbf{1}_{[s,t)}g](v)L(dv) \\
&= \int_{-\infty}^s g(v)M_d(dv) + \int_{-\infty}^s I^d_-[\mathbf{1}_{[s,t)}g](v)L(dv) \\
&\quad + \int_s^t I^d_-[\mathbf{1}_{[s,t)}g](v)L(dv).
\end{aligned} \tag{5.5}$$

By definition we know that $\int_{-\infty}^s g(v)L(dv)$ and $\int_{-\infty}^s I^d_-[\mathbf{1}_{[s,t)}g](v)L(dv)$ are $\mathcal{F}^{M_d}_s$-measurable since $\mathcal{F}^{M_d}_s = \mathcal{F}^L_s$ for $s \geq 0$. Therefore we obtain

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big\{iu\int_{-\infty}^t g(v)M_d(dv)\Big\}\Big|\mathcal{F}^{M_d}_s\Big] \\
= \; &\exp\Big\{iu\Big[\int_{-\infty}^s g(v)M_d(dv) + \int_{-\infty}^s I^d_-[\mathbf{1}_{[s,t)}g](v)L(dv)\Big]\Big\} \\
&\times\mathbb{E}\Big[\exp\Big\{iu\int_s^t I^d_-[\mathbf{1}_{[s,t)}g](v)L(dv)\Big\}\Big|\mathcal{F}^{M_d}_s\Big]
\end{aligned}$$

However due to the fact that

$$\operatorname{supp}(I^d_-[\mathbf{1}_{[0,w)}]) \cap [s,t) = \emptyset$$

for $w \in \mathbb{R}$, $s,t \in \mathbb{R}^+$ with $w \le s \le t$, the expression in the conditional expectation is independent of $\mathcal{F}_s^{M_d}$ and the unconditional expectation can be calculated by Theorem 2.7 of Rajput and Rosinski (1989):

$$\begin{aligned} & \mathbb{E}\Big[\exp\Big\{iu\int_s^t I_-^d[\mathbf{1}_{[s,t)}g](v)L(dv)\Big\}\Big|\mathcal{F}_s^{M_d}\Big] \\ = \; & \mathbb{E}\Big[\exp\Big\{iu\int_s^t I_-^d[\mathbf{1}_{[s,t)}g](v)L(dv)\Big\}\Big] = \exp\Big\{\int_s^t \psi\{u\cdot I_-^d[\mathbf{1}_{[s,t)}g](v)\}dv\Big\}. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} \int_{-\infty}^s I_-^d[\mathbf{1}_{[s,t)}g](v)L(dv) &= \int_{\mathbb{R}} \mathbf{1}_{(-\infty,s)}(v)I_-^d[\mathbf{1}_{[s,t)}g](v)L(dv) \\ &= \int_{\mathbb{R}} I^d\mathbf{D}_-^d\mathbf{1}_{(-\infty,s)}(v)I_-^d[\mathbf{1}_{[s,t)}g](v)L(dv) \\ &= \int_{\mathbb{R}} \mathbf{D}_-^d[\mathbf{1}_{(-\infty,s)}I_-^d[\mathbf{1}_{[s,t)}g]](v)M_d(dv) \\ &= \int_{-\infty}^s \mathbf{D}_-^d[\mathbf{1}_{(-\infty,s)}I_-^d[\mathbf{1}_{[s,t)}g]](v)M_d(dv) \end{aligned}$$

by applying Proposition 5.1 and Theorem 2.3. Also, we used in the last line that

$$\operatorname{supp}(\mathbf{D}_-^d[\mathbf{1}_{(-\infty,s)}I_-^d[\mathbf{1}_{[s,t)}g]]) \subseteq (-\infty, s).$$

Since $\mathcal{F}_s^{M_d} = \mathcal{F}_s^L$ for $s \ge 0$, the assertion follows. □

One of the main steps in the previous proof was based on

$$\operatorname{supp}(I_-^d[\mathbf{1}_{[0,w)}]) \cap [s,t) = \emptyset$$

for $w \in \mathbb{R}$, $s,t \in \mathbb{R}^+$ with $w \le s \le t$ which implied $\mathcal{F}_s^{M_d} = \mathcal{F}_s^L$ for $s \ge 0$. However, as discussed at the beginning of Section 3, the equality of the filtration no longer holds true if we condition on $s < 0$. Therefore we have to consider both possible cases in this situation starting with conditioning on $\mathcal{F}_s^{M_d}$.

Lemma 5.3. *For $d \in (-\frac{1}{2}, \frac{1}{2})$ let $g \in H^d$. Then equation (i) holds in distribution for all $u \in \mathbb{R}$ and $s,t \in \mathbb{R}$ with $s \le t$ and $s < 0$.*

Proof. The decomposition (5.5) no longer helps, since now

$$\operatorname{supp}(I_-^d[\mathbf{1}_{[0,s)}]) \cap [s,t) \neq \emptyset.$$

As consequence, the random variable $\int_{-\infty}^{s} I_-^d[\mathbf{1}_{[s,t)}g](v)L(dv)$ does no longer have to be $\mathcal{F}_s^{M_d}$-measurable and $\int_s^t I_-^d[\mathbf{1}_{[s,t)}g](v)L(dv)$ is not independent of $\mathcal{F}_s^{M_d}$ anymore. Therefore, choose $c > 0$ such that $s+c \geq 0$. Due to the stationarity of the M_d-increments (Proposition 2.2 (iii)) we obtain

$$\begin{aligned}
& \mathbb{E}\Big[\exp\Big\{iu\int_{-\infty}^{t} g(v)M_d(dv)\Big\}\Big|\mathcal{F}_s^{M_d}\Big] \\
\overset{d}{=} \; & \mathbb{E}\Big[\exp\Big\{iu\int_{-\infty}^{t+c} g(v-c)M_d(dv)\Big\}\Big|\mathcal{F}_{s+c}^{M_d}\Big] \\
= \; & \exp\Big\{iu\int_{-\infty}^{s+c} g(v-c)M_d(dv)\Big\} \\
& \times\mathbb{E}\Big[\exp\Big\{iu\int_{s+c}^{t+c} g(v-c)M_d(dv)\Big\}\Big|\mathcal{F}_{s+c}^{M_d}\Big] \qquad (5.6)
\end{aligned}$$

while $\mathcal{F}_{s+c}^{M_d} = \mathcal{F}_{s+c}^{L}$ holds again. Furthermore we get

$$\begin{aligned}
& \mathbb{E}\Big[\exp\Big\{iu\int_{s+c}^{t+c} g(v-c)M_d(dv)\Big\}\Big|\mathcal{F}_{s+c}^{M_d}\Big] \\
= \; & \mathbb{E}\Big[\exp\Big\{iu\int_{\mathbb{R}} \mathbf{1}_{[s+c,t+c)}(v)g(v-c)M_d(dv)\Big\}\Big|\mathcal{F}_{s+c}^{M_d}\Big] \\
= \; & \mathbb{E}\Big[\exp\Big\{iu\int_{\mathbb{R}} I^d[\mathbf{1}_{[s+c,t+c)}g(\cdot-c)](v)L(dv)\Big\}\Big|\mathcal{F}_{s+c}^{L}\Big] \\
= \; & \exp\Big\{iu\int_{-\infty}^{s+c} I^d[\mathbf{1}_{[s+c,t+c)}g(\cdot-c)](v)L(dv)\Big\} \\
& \times\mathbb{E}\Big[\exp\Big\{iu\int_{s+c}^{t+c} I^d[\mathbf{1}_{[s+c,t+c)}g(\cdot-c)](v)L(dv)\Big\}\Big|\mathcal{F}_{s+c}^{L}\Big]
\end{aligned}$$

Plugging this into equation (5.6) we see that the remaining expectation is independent of the filtration and by stationarity of the L-increments and Theorem 2.3 we get for the first part

$$
\begin{aligned}
& \exp\Big\{iu\int_{-\infty}^{s+c} g(v-c)M_d(dv) + iu\int_{-\infty}^{s+c} I^d[\mathbf{1}_{[s+c,t+c)}g(\cdot-c)](v)L(dv)\Big\} \\
= \;& \exp\Big\{iu\int_{-\infty}^{s+c} I^d[\mathbf{1}_{(-\infty,s+c)}g(\cdot-c)](v)L(dv)\Big\} \\
\times \;& \exp\Big\{iu\int_{-\infty}^{s+c} I^d[\mathbf{1}_{[s+c,t+c)}g(\cdot-c)](v)L(dv)\Big\} \\
\overset{d}{=} \;& \exp\Big\{iu\int_{-\infty}^{s} I^d[\mathbf{1}_{(-\infty,s+c)}g(\cdot-c)](v+c)L(dv)\Big\} \\
& \times\exp\Big\{iu\int_{-\infty}^{s} I^d[\mathbf{1}_{[s+c,t+c)}g(\cdot-c)](v+c)L(dv)\Big\} \\
= \;& \exp\Big\{iu\int_{-\infty}^{s} g(v)M_d(dv) + iu\int_{-\infty}^{s} I^d[\mathbf{1}_{[s,t)}g](v)L(dv)\Big\}.
\end{aligned}
$$

While for the conditional expectation, we obtain due to independent L-increments

$$
\begin{aligned}
& \mathbb{E}\Big[\exp\Big\{iu\int_{s+c}^{t+c} I^d[\mathbf{1}_{[s+c,t+c)}g(\cdot-c)](v)L(dv)\Big\}\Big|\mathcal{F}^L_{s+c}\Big] \\
= \;& \mathbb{E}\Big[\exp\Big\{iu\int_{s+c}^{t+c} I^d[\mathbf{1}_{[s+c,t+c)}g(\cdot-c)](v)L(dv)\Big\}\Big] \\
= \;& \exp\Big\{\int_{s+c}^{t+c} \psi\{u\cdot I^d[\mathbf{1}_{[s+c,t+c)}g(\cdot-c)](v)\}dv\Big\} \\
= \;& \exp\Big\{\int_{s}^{t} \psi\{u\cdot I^d[\mathbf{1}_{[s,t)}(\cdot-c)g(\cdot-c)](v+c)\}dv\Big\} \\
= \;& \exp\Big\{\int_{s}^{t} \psi\{u\cdot I^d[\mathbf{1}_{[s,t)}g](v)\}dv\Big\}.
\end{aligned}
$$

Putting everything together we obtain the assertion. □

We finally need to condition on $\mathcal{F}^L_s$, $s<0$, to obtain the last part of our main theorem.

Lemma 5.4. *For $d \in (-\frac{1}{2},\frac{1}{2})$ let $g \in H^d$. Then equation (ii) holds in the $L^2(\Omega)$-sense for all $u \in \mathbb{R}$ and $s,t \in \mathbb{R}$ with $s \le t$ and $s < 0$.*

Proof. We start with the decomposition (5.5): per definition, the random variable $\int_{-\infty}^{s} I^d_-[\mathbf{1}_{[s,t]}g](v)L(dv)$ is $\mathcal{F}^L_s$-measurable. Using

$$
\mathrm{supp}(I^d_-[\mathbf{1}_{(-\infty,s)}g]) \cap [s,\infty) = \emptyset
$$

and Theorem 2.3 we obtain

$$\int_{-\infty}^{s} g(v)M_d(dv) = \int_{\mathbb{R}} I_-^d[\mathbf{1}_{(-\infty,s)}g](v)L(dv) = \int_{-\infty}^{s} I_-^d[\mathbf{1}_{(-\infty,s)}g](v)L(dv)$$

and conclude that $\int_{-\infty}^{s} I_-^d[\mathbf{1}_{[s,t)}g](v)L(dv)$ is $\mathcal{F}_s^L$-measurable as well. Furthermore we have

$$\mathbb{E}\Big[\exp\Big\{iu\int_s^t I_-^d[\mathbf{1}_{[s,t)}g](v)L(dv)\Big\}\Big|\mathcal{F}_s^L\Big] = \mathbb{E}\Big[\exp\Big\{iu\int_s^t I_-^d[\mathbf{1}_{[s,t)}g](v)L(dv)\Big\}\Big]$$

and can carry out the rest of the proof analogously to the proof of Lemma 5.2. □

Proof of Theorem 3.1. Putting Lemma 1-3 together we obtain the results regarding the conditional characteristic functions. The specific representation for $\mathbf{D}_-^d[\mathbf{1}_{(-\infty,s)}I_-^d[\mathbf{1}_{[s,t)}g]]$ follows from Proposition 5.1. □

6. Outlook

Although we considered only univariate MvN-fLps, the obtained results on conditional characteristic functions can be straightforwardly extended to a multivariate setting as considered eg. in Marquardt (2007).

In addition, conditional characteristic functions can be used to price derivatives in fractional market settings based on MvN-fLps. However, in general, markets driven by fractional Lévy processes (whether of the MvN- or the MG-type) might allow for arbitrage as in the case of fractional Brownian motion (eg. Cheridito (2003) or Bender, Sottinen and Valkeila (2007)). Although there exists already a full characterization of when MvN-fLps are semimartingales (cf. Bender, Lindner and Schicks (2011)), this only allows for an equivalence of the existence of an equivalent martingale measure and the exclusion of arbitrage - it does not yet rule out arbitrage itself. In fact, as there are many similarities between fractional Brownian motion and MvN- or MG-fLps, one might conjecture that these markets even allow for arbitrage opportunities when the driving power is such a semimartingale. However, in the literature there exist several solutions to this general modeling issue (eg. introduction of transaction costs in Guasoni, Rásonyi and Schachermayer 2008; 2010) or restriction of trading strategies in Bender, Sottinen and Valkeila (2008) or Tikanmäki and Mishura (2011)). Therefore our results on conditional characteristic functions of MvN-fLps might also be useful in such situations.

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